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Invariants of the Dirichlet/Voronoi Tilings of Hyperspheres in Rn and their Dual Delone/Delaunay Graphs

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Preface

This collection is the Second part of the Proceedings of the Fifth International Conference on Analytic Number Theory and Spatial Tessellations which was held in Kyiv on September 16–20, 2013. This was the fifth Kyiv International Conference devoted to the present-day development of George Voronoi's scientific results. Its program covered all the main items of Voronoi's scientific achievements both in pure mathematics which served as the impetus for the new domains of science, and the applications of Voronoi diagrams in natural sciences.

The conference organizers planned to publish the Proceedings in 2014 under the traditional title *Voronoi's Impact on Modern Science*. Some part of the papers was prepared before the conference, but most of them had to be sent to the end of 2013. However, due to certain circumstances related to the political situation in our country, in a state of uncertainty and inability to predict the future development of the situation, as well as because of the lack of funds for the publication of Proceedings, we had to postpone it for a while.

It should be noted that the authors who had sent their materials for printing, reacted with understanding to the situation that had arisen and were patiently waiting for the realization of the collection publication, and even offered to publish some part of the papers in the Lithuanian scientific journal *Šiauliai Mathematical Seminar*.

We are grateful to the Lithuanian colleagues for their help in such a critical situation, as well as to Professor Jörn Steuding, who offered this option. Thereby, the proposed collection is the second part (the first part is being published in Kyiv) of the Proceedings of the 5th International Conference on Analytic Number Theory and Spatial Tessellations.

We hope that the interest in Voronoi's ideas will contribute to the further development of science and that the tradition of the Kyiv international Voronoi conferences will continue.

We believe that this issue will be another step in the development of the fruitful ideas of George Voronoi.

Mykola Pratsiovytyi
Editor of the Proceedings

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INVARIANTS OF THE DIRICHLET/VORONOI TILINGS OF HYPERSPHERES IN $\mathbb{R}^N$ AND THEIR DUAL DELONE/DELAUNAY GRAPHS

FRANCESC ANTÓN CASTRO

Abstract. In this paper, we are addressing the geometric and topological invariants that arise in the exact computation of the Delone (Delaunay) graph and the Dirichlet/Voronoi tiling of n-dimensional hyperspheres using Ritt-Wu's algorithm. Our main contribution is a methodology for automated derivation of geometric and topological invariants of the Dirichlet tiling of $N+1$-dimensional hyperspheres and its dual Delone graph from the invariants of the Dirichlet tiling of N-dimensional hyperspheres and its dual Delone graph (starting from $N=3$).

Key words and phrases: Delaunay graph of hyperspheres, geometric invariants, Ritt-Wu characteristic set method, Voronoi diagram of hyperspheres, topological invariants.

2010 Mathematics Subject Classification: 14Q15, 65D18, 51M15, 86A30.

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1. Introduction

In this paper, we use the words Delaunay or Delone with the same meaning (to reflect the orthography of their inventor). Voronoi diagrams have been a central topic in research in computational geometry since their inception [11, 7, 4, 5, 13], and they have many applications in different scientific and engineering disciplines [14, 23]. However, generalized Voronoi diagrams, and especially the Voronoi diagram of spheres, have not been explored sufficiently [17]. With recent scientific discoveries in biology and chemistry, Voronoi diagrams of spheres have become more important for representing and analysing the molecular 3D structure and surface [24], the structure of the protein [15], etc.

Gavrilova provided an early work on generalised Voronoi diagrams in her doctoral thesis [9], and subsequent work by Gavrilova and Rokne on topology updating of the kinematic Voronoi diagram of hyperspheres [8]. Gavrilova was the first one to provide an explicit algorithm for the computation of the vertices of the Voronoi diagram of spheres [10]. The degree of their predicate for hyperspheres in d-dimensional space was $2(d+1)$ in the variables defining the spheres, thus 8 in the case of spheres in $3D$ [8]. They suggest using Newton's method for the computation of the Delaunay empty sphere criterion, thus not an exact computation. Our predicate is computed exactly evaluating a degree 6 polynomial in 3D. We have found the formulas analogous to formulas (7), (8), (9) and (10) presented in [8] automatically using Wu's algorithm [28]: they constitute the Ritt-Wu characteristic set [28] for the polynomial set corresponding to the Voronoi vertex of four spheres. In their work, Gavrilova and Rokne do not actually state explicitly that these were invariants nor their geometric interpretation nor all their algebraic relationships (syzygies or rewriting rules).

Nishida and Sugihara [21] and Nishida et al. [22] extended the results of Gavrilova and Rokne [8] by providing the topological structure of the Voronoi diagram of hyperspheres in d-dimensional space using finite precision arithmetic (double). They prove in [21] that they need only $2d+4$ times longer bits for exact computation than the bits used for the input. They exhibit in the formulas (38), (39), (40) and (41) in [21] linear relations between the grouping of terms in the function they evaluate in floating-point arithmetic. However, they do not actually state explicitly that these were invariants nor their geometric interpretation nor all their non-linear algebraic relationships (syzygies or rewriting rules).

Moreover, all of the current research efforts did not provide the exact method for computing the Delaunay graph (or quasi-triangulation) of hyperspheres and their invariants. The dual graph of the Voronoi diagram, which Anton [2, 3] names "Delaunay graph", is the graph corresponding to the simplicial complex that was later named "quasi-triangulation" and studied in Kim et al. [18, 16]. In this paper, we are addressing the geometric and topological invariants that arise in the exact computation of the Delone graph and the Dirichlet/Voronoi tiling of n-dimensional hyperspheres using Ritt-Wu's algorithm rather than Gröbner bases, because the complexity of Ritt-Wu's algorithm is simply exponential [20], while the complexity of Gröbner bases is doubly exponential. Our main contribution is first a methodology for automated derivation of geometric and topological invariants of the Dirichlet tiling of $n+1$-dimenional hyperspheres and its dual Delone graph from the invariants of the Dirichlet tiling of n-dimensional hyper-spheres and its dual Delone graph (starting from $n=3$). The 3D case was treated in Anton et al. [1]. The reader is invited to refer to that paper for an explanation of the previous work and the 3D case.

The exact knowledge of the Delaunay graph for curved objects may sound like a purely theoretical knowledge, but in fact, it is central in some practical applications. These applications include material science, metallography, spatial analyses and VLSI layout. The exact knowledge of the neighbourliness among molecules is central to the prediction of the formation of particle aggregates in material science (the Kolmogorov equation [19]). Finally, the exact computation of the Delaunay graph of hyperspheres participates to the recent move in the development of numerical and simulation software as well as computer algebra systems to exact systems [6].

This paper is organised as follows. In Section 1, we review the preliminaries: the Voronoi diagram and Delaunay graph of hyperspheres. In Section 2, we present Wu's method. In Section 3, we present a novel method of computation of the topological and geometric invariants of the Voronoi Diagram and Delaunay graph of hyperspheres in $\mathbb{R}^4$ from the corresponding invariants in $\mathbb{R}^3$ using Wu's method. Section 4 is devoted to the generalization of the results presented in Section 3 to obtaining the invariants in $\mathbb{R}^{N+1}$ from those in $\mathbb{R}^N$. Finally, in Section 5, we conclude the paper.

2. Preliminaries

Voronoi diagrams are irregular tessellations of space, where space is continuous and structured by discrete objects [23]. The Voronoi tessellation of a set of sites is a decomposition of the space into proximal regions (one for each site). Sites were points for the first historical Voronoi diagrams [25, 26, 27], but, in this paper, we will explore sets of hyperspheres in $\mathbb{R}^N$. The proximal region corresponding to one site (i.e. its Voronoi region) is the set of points of the space that are closer to that site than to any other site of the set of sites [23]. We will recall now the formal definitions of the Voronoi diagram and of the Delaunay graph. For this purpose, we need to recall some basic definitions.

Definition 1. Let M be an arbitrary set. A metric on M is a mapping $\mathrm{d}: M \times M \to \mathbb{R}_+$ such that, for any elements a, b, and c of M, the following conditions are fulfilled: $\mathrm{d}(a,b)=0 \Leftrightarrow a=b$, $\mathrm{d}(a,b)=\mathrm{d}(b,a)$, and $\mathrm{d}(a,c) \leqslant \mathrm{d}(a,b)+\mathrm{d}(b,c)$. (M,d) is then called a metric space, and $\mathrm{d}(a,b)$ is the distance between a and b.

Let $M=\mathbb{R}^N$, and δ denote a distance between points. Let $\mathcal{S}=\{s_1,\ldots,s_m\}$, $m \geqslant 2$, be a set of m different subsets of M, which we call sites. The distance between a point x and a site $s_i \in \mathcal{S}$ is defined as $d(x,s_i)=\inf_{y\in s_i}\{\delta(x,y)\}$.

Definition 2. For $s_i, s_j \in \mathcal{S}, s_i \neq s_j$, the bisector $\mathrm{B}(s_i,s_j)$ of s_i with respect to s_j is: $\mathrm{B}(s_i,s_j)=\{x \in M | d(x,s_i)=d(x,s_j)\}$.

Definition 3. For $s_i, s_j \in \mathcal{S}, s_i \neq s_j$, the influence zone $\mathrm{D}(s_i,s_j)$ of s_i with respect to s_j is: $\mathrm{D}(s_i,s_j)=\{x \in M | d(x,s_i) < d(x,s_j)\}$.

Definition 4. The Voronoi region $\mathrm{V}(s_i,\mathcal{S})$ of $s_i \in \mathcal{S}$ with respect to the set $\mathcal{S}$ is: $\mathrm{V}(s_i,\mathcal{S})=\cap_{s_j\in\mathcal{S},s_j\neq s_i}\mathrm{D}(s_i,s_j)$.

Definition 5. The Voronoi diagram of $\mathcal{S}$ is the union $\mathrm{V}(\mathcal{S})=\cup_{s_i\in\mathcal{S}}\partial\mathrm{V}(s_i,\mathcal{S})$ of all region boundaries.

Definition 6. The Delaunay graph $DG(\mathcal{S})$ of $\mathcal{S}$ is the dual graph of $V(\mathcal{S})$ defined as follows:

- the set of vertices of $DG(\mathcal{S})$ is $\mathcal{S}$;
- for each $(N-1)$-dimensional facet of $V(\mathcal{S})$ that belongs to the common boundary of $V(s_i, \mathcal{S})$ and of $V(s_j, \mathcal{S})$ with $s_i, s_j \in \mathcal{S}$ and $s_i \neq s_j$, there is an edge of $DG(\mathcal{S})$ between s_i and s_j and reciprocally;
- for each vertex of $V(\mathcal{S})$ that belongs to the common boundary of $V(s_{i_1}, \mathcal{S}), \ldots, V(s_{i_{N+2}}, \mathcal{S})$ for all $k \in \{1, \ldots, N+2\}, s_{i_k} \in \mathcal{S}$ all distinct, there exists a complete graph K_{N+2} between the s_{i_k}, $k \in \{1, \ldots, N+2\}$, and reciprocally.

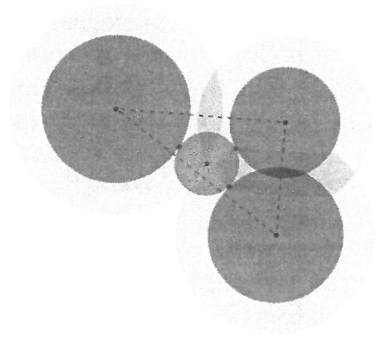

Figure 1. The Voronoi vertex is at the same distance with respect to its three defining circles (in blue). The circumcircle externally tangent to these four defining spheres is represented in green. The Delone graph is represented in dashed green lines.

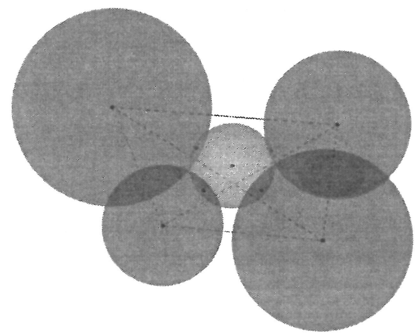

Figure 2. The Voronoi vertex is at the same distance with respect to its four defining spheres (in blue). The circumsphere externally tangent to these four defining spheres is represented in green. The Delone graph is represented in dashed green lines.

The Delaunay graph satisfies the Delonay empty circumsphere criterion: since a Voronoi vertex has at least $N+1$ nearest neighbors, it must be the center of an hypersphere, that is tangent to its nearest neighboring hypersphere, and whose interior does not contain any point of any nearest neighboring hypersphere (see Figure 1 for the 2D case of the Voronoi diagram of circles, and Figures 2 and 3 for the 3D case of the Delone graph empty circumsphere criterion for spheres). Notice that the Delaunay empty circumsphere might be externally tangent to all the neighboring hypersphere or internally tangent to all the neighboring hyperspheres, or internally tangent to some of them and externally tangent to the other nearest neighbors (see Figures 4, 5, 6 and 7).

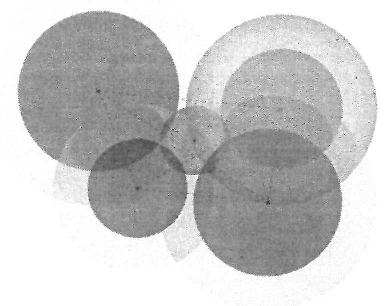

Figure 3. The Voronoi vertex is at the same distance with respect to its four defining spheres (in blue). The circumsphere externally tangent to the four defining spheres is represented in green. The Voronoi vertex is the center of four spheres (in red) that are expansions of the four defining spheres. The Delone graph is represented in dashed green lines.

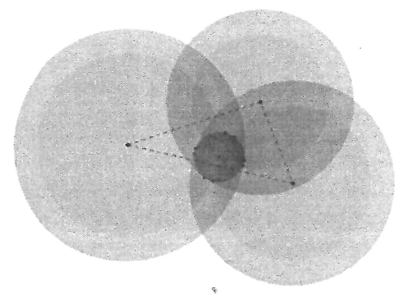

Figure 4. The Voronoi vertex is at the same distance with respect to its three defining circles (in blue). The circumcircle internally tangent to these four defining spheres is represented in green. The Delone graph is represented in dashed green lines.

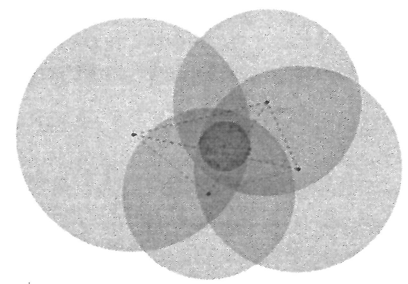

Figure 5. The Voronoi vertex is at the same distance with respect to its four defining spheres (in blue). The circumsphere internally tangent to these four defining spheres is represented in green. The Delone graph is represented in dashed green lines.

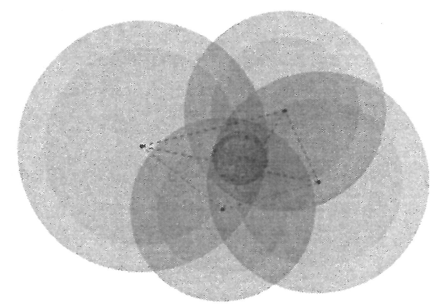

Figure 6. The Voronoi vertex is at the same distance with respect to its four defining spheres (in blue). The circumsphere internally tangent to the four defining spheres is represented in green. The Voronoi vertex is the center of four spheres (in red) that are shrinkages of the four defining spheres. The Delone graph is represented in dashed green lines.

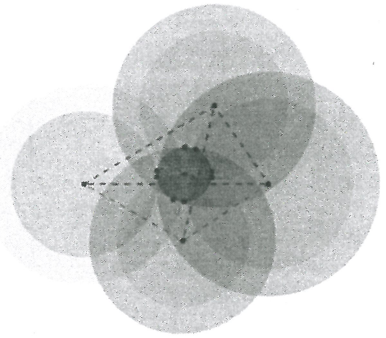

Figure 7. The Voronoi vertex is at the same distance with respect to its four defining spheres (in blue). The circumsphere (internally or externally) tangent to the four defining spheres is represented in green. The Voronoi vertex is the center of four spheres (in red) that are either expansions or shrinkages of the four defining spheres. The Delone graph is represented in dashed green lines.

3. Wu algorithm

Let $\mathcal{K} = \mathbb{Q}$ be the field of rational numbers, $\mathbb{X} = \{x_1, \ldots, x_n\}$ a set of variables, $\mathcal{K}[\mathbb{X}]$ be the set of polynomials in the variables of $\mathbb{X}$ with coefficients in $\mathcal{K}$. If not otherwise stated, the order of the monomials composing a polynomial will be taken as the lexicographic monomial ordering where $1 \prec x_1 \prec x_2 \prec \ldots \prec x_n$. The universal field $\mathcal{E}$ over $\mathcal{K}$ is an algebraically closed field containing an infinite number of indeterminates, or, more simply, a projective space over an algebraically closed field.

Definition 7 ([28]). For any set of polynomials $\mathbb{P} \subset \mathcal{K}[\mathbb{X}]$, $\mathrm{Zero}(\mathbb{P}) = \{x \in \mathcal{E}^n | \forall P \in \mathbb{P}, P(x) = 0\}$ is called a variety. For a set of polynomials $\mathbb{P}$ and a polynomial D, we define $\mathrm{Zero}(\mathbb{P}/D) = \mathrm{Zero}(\mathbb{P}) \setminus \mathrm{Zero}(\{D\})$, called a quasi-algebraic variety.

The aim of Wu's method is to determine the decomposition of quasi-algebraic varieties into irreducible components and the dimensions of these irreducible components. For example, $\mathrm{Zero}(\{xy\})$ in the three-dimensional projective space corresponds to the union of two irreducible components of dimension 2: the projective plane $\mathrm{Zero}(\{x\})$ and the projective plane $\mathrm{Zero}(\{y\})$. Likewise, Wu's method al-

lowed us to prove that the generalized v-offset to a sphere of center Π and of radius r (i.e., the set of centers of spheres of radius v that are tangent to that sphere) is the union of two irreducible components of dimension 2: the sphere of center Π and of radius $r + v$, and the sphere of center Π and of radius $|r - v|$. The difference between the generalized v-offset and the true v-offset to a given sphere is that the true v-offset is the set of centers of circles of radius v that are externally tangent to that given sphere while the generalized v-offset is the set of centers of circles of radius v that are internally or externally tangent to that given sphere. If $v \leqslant r$, than the generalized and the true offsets coincide while otherwise $(v > r)$, the true offset includes only the bigger sphere of radius $r + v$.

Definition 8 ([28]). Let $P \in \mathcal{K}[\mathbb{X}]$ be a polynomial. The class of P, denoted by $\mathrm{cls}(P)$ is the c such that x_c is the largest variable that occurs in P. If $\mathrm{cls}(P) = c$, then x_c is called the leading variable and denoted by $\mathrm{lvar}(P)$, the highest degree monomial of P as a univariate polynomial in $\mathrm{lvar}(P)$ is called the leading monomial, and its coefficient is called the initial of P and denoted by $\mathrm{init}(P)$.

Typically, the polynomial D whose variety is subtracted from a polynomial set will be a product of initials corresponding to geometric invariants that correspond to special cases of the geometric problem at hand. In all this paper, we mean by geometric invariant a quantity that is invariant under some geometric transformation, and we mean by topological invariant a quantity or a quality that is invariant by any continuous transformation.

Definition 9 ([28]). A polynomial P_1 has higher ordering than a polynomial P_2, denoted as $P_2 \prec P_1$, if either $\mathrm{cls}(P_1) > \mathrm{cls}(P_2)$, or $c = \mathrm{cls}(P_1) = \mathrm{cls}(P_2)$ and $\deg(P_1, x_c) > \deg(P_2, x_c)$ (where deg denotes the degree of the polynomial). If none of two polynomials has higher ordering than the other, they are said to have the same rank, denoted as $P_1 \sim P_2$.

Definition 10 ([28]). A polynomial Q is reduced with respect to P if $\mathrm{cls}(P) = c > 0$ and $\deg(Q, x_c) < \deg(P, x_c)$. A sequence of non-zero polynomials $\mathcal{A} : A_1, A_2, \ldots, A_r$ is a triangular set if either $r = 1$ or $\mathrm{cls}(A_1) < \mathrm{cls}(A_2) < \ldots < \mathrm{cls}(A_r)$. A triangular set is called an ascending chain, or simply a chain, if A_j is reduced with respect to A_i for $i < j$. For a chain $\mathcal{A}$, we denote $\mathbb{I}_{\mathcal{A}}$ as the product of the initials of the polynomials in $\mathcal{A}$.

Definition 11 ([28]). Let $\mathcal{A}' : A'_1, A'_2, \ldots, A'_{r'}$ and $\mathcal{A}'' : A''_1, A''_2, \ldots, A''_{r''}$ be two (ascending) chains. $\mathcal{A}'$ is said to be of lower ordering than $\mathcal{A}''$, denoted as $\mathcal{A}' \prec \mathcal{A}''$, if either there is some k such that $A'_1 \sim A''_1 \ldots A'_{k-1} \sim A''_{k-1}$, while $A'_k \prec A''_k$; or $r' > r''$ and $A'_1 \sim A''_1 \ldots A'_{r''} \sim A''_{r''}$.

Lemma 1 ([28]). *A sequence of (ascending) chains steadily lower in ordering is finite.*

Definition 12 ([28]). A basic set of a polynomial set $\mathbb{P}$ is any chain of lowest ordering contained in $\mathbb{P}$. A polynomial Q is called reduced with respect to a chain $\mathcal{A}$ if Q is reduced with respect to all the polynomials in $\mathcal{A}$.

LEMMA 2 ([28]). *Let $\mathcal{A}$ be a basic set of a polynomial set $\mathbb{P}$. If P is reduced with respect to $\mathcal{A}$, then a basic set of $\mathbb{P} \cup P$ is of lower ordering than that of $\mathbb{P}$.*

Let F and G be non-zero polynomials with $c = cls(F)$ and $I = \text{init}(F)$. Either G is reduced with respect to F (which means that $\deg(G, x_c) < \deg(F, x_c)$), or $\deg(G, x_c) \geqslant \deg(F, x_c)$, and then it is possible to divide G by F as univariate polynomials in x_c. Indeed, let $k = \deg(G, x_c) - \deg(F, x_c)$, $k' = \deg(G, x_c)$, and I' be the coefficient of $x_c^{k'}$ in G, then $\deg\left(IG - I'x_c^k F\right) < k'$. Therefore, in a finite number of steps $s \leqslant k+1$, we get that $I^s G = QF + R$, where Q and R are polynomials in $\mathcal{K}[\mathbb{X}]$ with R reduced with respect to F. R is uniquely determined and called the pseudo-remainder of G with respect to F and denoted as $R = \text{prem}(G, F)$.

It is therefore trivial to generalize this Euclidean division to the case of a triangular system: the result of the division of a polynomial G with respect to the polynomials of the triangular set $\mathcal{A} : A_1, A_2, \ldots, A_r$ is obtained by repeated division of G by A_1, A_2, …, A_r. Thus, we get the division formula: $JG = \sum_i Q_i A_i + R$, where J is a polynomial in $\mathcal{K}[\mathbb{X}]$, R is reduced with respect to all the polynomials in the triangular set $\mathcal{A}$, the polynomials Q_i are in $\mathcal{K}[\mathbb{X}]$ and R is called the pseudo-remainder of G with respect to $\mathcal{A}$, and denoted as $R = \text{prem}(G, \mathcal{A})$ [28].

This leads to Wu's algorithm for producing the decomposition of a variety into irreducible varieties (corresponding to irreducible polynomial sets). Starting with a polynomial set $\mathbb{P}_0 = \mathbb{P}$, one should select a basis $\mathcal{B}_0$ of $\mathbb{P}_0$, and then compute the set $\mathbb{R}_0$ of non-zero pseudo-remainders of polynomials of $\mathbb{P}_0 \setminus \mathcal{B}_0$ with respect to $\mathcal{B}_0$. Then, let $\mathbb{P}_1 = \mathbb{P}_0 \cup \mathbb{R}_0$. Then, one should compute a basis set $\mathbb{B}_1$ in $\mathbb{P}_1$. By Lemma 2, $\mathcal{B}_1$ is of lower ordering than $\mathcal{B}_0$. After this, one should compute the set $\mathbb{R}_1$ of non-zero pseudo-remainders of polynomials of $\mathbb{P}_1 \setminus \mathcal{B}_1$ with respect to $\mathcal{B}_1$. Therefore, such a process has a finite number of steps, and the final result is a basic set $\mathcal{B}_m = \mathcal{C}$, such that the corresponding set of non-zero pseudo-remainders $\mathbb{R}_m$ is the empty set and $\text{prem}(\mathbb{P}, \mathcal{C}) = \{0\}$. Thus, for each chain that can be obtained in such a way, $\text{Zero}(\mathbb{P}) \subseteq \text{Zero}(\mathcal{C})$. Therefore, one can obtain the variety of $\mathbb{P}$ as a union of quasi-projective varieties, where each algebraic variety is irreducible and the algebraic varieties being subtracted correspond to degenerate cases expressed as geometric invariants. Any chain $\mathcal{C}$ obtained by applying Wu's algorithm is called a characteristic set [28].

4. From the invariants of the Voronoi diagram of spheres in $\mathbb{R}^3$ to those of the Voronoi diagram of hyperspheres in $\mathbb{R}^4$

Starting from $N = 3$, we need to change the following polynomial set [1]. Since the generalized v-offset of a sphere centered on $(a, b, c)^T$ and of radius s is the union of two concentric spheres centered on $(a, b, c)^T$ and of radii $s + v$ and $|s - v|$, we need to consider hyperspheres of radii that increase of v and that decrease of v and thus, the justification for the $\pm v$ in the following polynomial set. Thus, we can state that a Voronoi vertex is a zero of one of the following polynomial sets (i.e., a point on which all the polynomials in one of the following polynomial sets $\mathcal{II}$ evaluate to 0), where we have set the origin of the coordinate system at the center of the smallest hypersphere and subtracted from the radius of each hypersphere the radius of the smallest one:

$$\mathcal{II} : \quad \left\{ \begin{array}{c} x^2 + y^2 + z^2 - (v)^2 \\ (x - a')^2 + (y - b')^2 + (z - c')^2 - (s' \pm v)^2 \\ (x - d')^2 + (y - e')^2 + (z - f')^2 - (t' \pm v)^2 \\ (x - g')^2 + (y - h')^2 + (z - i')^2 - (u' \pm v)^2 , \end{array} \right.$$

where $(a', b', c') = (d, e, f) - (a, b, c)$, $(d', e', f') = (g, h, i) - (a, b, c)$, $(g', h', i') = (j, k, l) - (a, b, cr)$, $s' = s - r$, $t' = t - r$ and $u' = u - r$.

This simplification shows that the apparent 16 possible configuration cases of four spheres (corresponding to the 16 possible cases of system $\mathcal{I}$ in [1]) are not linearly independent, but pair up two by two, since the nature of the irreducible component of the first generalized offset (expansion or retraction) does not have any influence on the polynomial sets $\mathcal{II}$ whose common roots are the Voronoi vertices of the four spheres. Now we consider the system where all the vs are preceded by a $+$ sign. The same basic set shows that by subtracting the equation of the first generalized offset from the equations of the second, third and fourth generalized offsets, we get an equivalent polynomial set composed of a single quadratic polynomial s_1 of a sphere generalized offset and three linear polynomials p_1, p_2 and p_3:

$$\left\{ \begin{array}{c} x^2 + y^2 + z^2 - (v)^2 \\ -2a'x - 2b'y - 2c'z - 2s'v + (a'^2 + b'^2 + c'^2 - s'^2) \\ -2d'x - 2e'y - 2f'z - 2t'v + (d'^2 + e'^2 + f'^2 - t'^2) \\ -2g'x - 2h'y - 2i'z - 2u'v + (g'^2 + h'^2 + i'^2 - u'^2) \end{array} \right.$$

In the remainder of the paper, we will denote any hypersphere either by its name, or its name followed in parentheses by its center and its radius (e.g. $S_1\left((a, b, c)^T, r\right)$ denotes the hypersphere centered at $(a, b, c)^T$ and of radius r. A true Voronoi vertex of five hyperspheres in $\mathbb{R}^4$ S_1, S_2, S_3, S_4 and S_5 is the intersection of five true[6] v-offsets to S_1, S_2, S_3, S_4 and S_5. This is indeed the case because even if a point is in the interior of the disk bounded by a sphere (i.e. inside the sphere but not on the sphere), it has a positive distance with respect to the sphere. Therefore, we need to consider both positive offsets (expansions) of spheres and negative offsets (contractions) of spheres in the equations of the generalized offsets of spheres. The offset to a geometric object is defined as the locus of points at the offset parameter distance from the given geometric object.

Therefore, a Voronoi vertex is a solution of one of the following systems $\mathcal{I}$ of

[6] which, in this case, are algebraic sets

polynomial equations:

$$\begin{cases} s_1 &= x^2 + y^2 + z^2 + w^2 - (v)^2, \\ pr_1 &= -2ax - 2by - 2cz - 2jw \pm 2sv + (a^2 + b^2 + c^2 + j^2 - s^2), \\ pr_2 &= -2dx - 2ey - 2fz - 2kw \pm 2tv + (d^2 + e^2 + f^2 + k^2 - t^2), \\ pr_3 &= -2gx - 2hy - 2iz - 2lw \pm 2uv + (g^2 + h^2 + i^2 + l^2 - u^2), \\ pr_4 &= -2mx - 2ny - 2oz - 2pw \pm 2qv + (m^2 + n^2 + o^2 + p^2 + q^2, \end{cases}$$

where w is the last (fourth) coordinate of the Voronoi vertex and we have set the origin of the coordinate system at the center of the smallest hypersphere and subtracted from the radius of each hypersphere the radius of the smallest one.

In this system, all polynomials have the same class: if we assume a variable ordering $v \prec w \prec z \prec y \prec x$, all the polynomials above have w has higher ordering variable (i.e., class) and their degree in the class is 2 for the equation of the smallest hypersphere s_1 and 1 for the equations of the radical planes p_1, p_2, p_3 and p_4. Thus, s_1 has a higher ordering than p_1, p_2, and p_3 and p_4. Therefore, it is possible to divide s_1, p_1, p_2, p_3 or p_4 by p_1, p_2, p_3 or p_4. We start with $\mathbb{P}_0 = \{s_1, p_1, p_2, p_3, p_4\}$ and an ascending chain $\mathcal{A}_0 = p_1$. An ascending chain is obtained from $\mathbb{P}_0$ by repeated division of the polynomials that are not already in the ascending chain, adding all non-zero pseudo-remainders to the polynomial set $\mathbb{P}_0$, and adding the non-zero pseudo-remainder with lowest ordering to the chain $\mathcal{A}_0$. The pseudo-remainders of the division by $\mathcal{A}_0$ are polynomials in y, z, w and v. The first pseudo-remainder that has the lowest ordering is $pr_1 = \text{prem}\,(p_2, p_1)$. It can be added to $\mathcal{A}_0$. However, $pr_2 = \text{prem}\,(p_3, p_1)$, $pr_3 = \text{prem}\,(p_4, p_1)$ and $pr_4 = \text{prem}\,(s_1, p_1)$ cannot be added to $\mathcal{A}_0$, since they do not have lower ordering than pr_1. The next non-zero pseudo-remainder that is added to the chain $\mathcal{A}_1 = p_1, pr_1$ is $pr_5 = \text{prem}\,(pr_2, pr_1)$, which is a polynomial in v, w and z. The other non-zero pseudo-remainders at this class level are cannot be added to the chain (because they do not have lower ordering): $pr_6 = \text{prem}\,(pr_3, pr_1)$ and $pr_7 = \text{prem}\,(pr_4, \mathcal{A}_1)$. The next non-zero pseudo-remainder that is added to the chain $\mathcal{A}_2 = p_1, pr_1, pr_5$ is $pr_8 = \text{prem}\,(pr_6, pr_5)$, which is a polynomial in v and w. The other non-zero pseudo-remainder cannot be added to the chain: $pr_9 = \text{prem}\,(pr_7, \mathcal{A}_2)$. We obtain a basis set when the lowest ordering non-zero pseudo-remainder is a single-valued polynomial. In this case, this is achieved after the following non-zero pseudo-remainder is computed: $pr_{10} = \text{prem}\,(pr_9, \mathcal{A}_3)$, which is a quadratic polynomial in v. However, from all the computed non-zero pseudo-remainders, we select polynomials with smallest ordering, Newton polytope and computer representation corresponding to a change of monomial order. Thus, our basic set is $\mathcal{C} : C_1, C_2, C_3, C_4, C_5$, where

$$\begin{cases} C_1 &= Jv^2 + Kv + L, \\ C_2 &= Ax + Hv + I, \\ C_3 &= -Ay + Ev + F, \\ C_4 &= Az + Bv + C, \\ C_5 &= -Aw + Nv + O. \end{cases}$$

It is a characteristic set of $\mathbb{P}$, since the class is strictly increasing along the ascending chain, and every polynomial C_j occurring after a polynomial C_i (with $j > i$) is reduced with respect of C_j. Since the univariate polynomial is of degree 2 in v, all the pseudo-remainders of polynomials by $\mathcal{A}$ will be univariate polynomials of degree at most 1 in v. The offset variable can be computed by solving the quadratic equation $Jv^2+Kv+L = 0$, which has no solution if $K^2 < 4JL$, one solution $v = \frac{K}{2J}$ if $K^2 = 4JL$, and two solutions $v = \frac{K \pm \sqrt{K^2 - 4JL}}{2J}$ if $K^2 > 4JL$. Using Wu's algorithm and the monomial order where each one of the invariants of the $\mathbb{R}^3$ case and its minors and j, k, l, m, n, o, p, q have lower ordering than $a, b, c, d, e, f, g, h, i, s, t, u$, we derived the coefficients of the monomials of the polynomials in the preceding basic set:

$$A = \begin{vmatrix} & A' & & j \\ & & & k \\ & & & l \\ m & n & o & p \end{vmatrix}, \quad A' = -2 \begin{vmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & i' \end{vmatrix},$$

$$B = \begin{vmatrix} & B' & & j \\ & & & k \\ & & & l \\ m & n & -q & p \end{vmatrix}, \quad B' = 2 \begin{vmatrix} a' & b' & -s' \\ d' & e' & -t' \\ g' & h' & -u' \end{vmatrix},$$

$$C = \begin{vmatrix} & & C' & j \\ & & & k \\ & & & l \\ m & n & m^2 + n^2 + o^2 - q^2 & p \end{vmatrix},$$

$$C' = \begin{vmatrix} a' & b' & a'^2 + b'^2 + c'^2 - s'^2 \\ d' & e' & d'^2 + e'^2 + f'^2 - t'^2 \\ g' & h' & g'^2 + h'^2 + i'^2 - u'^2 \end{vmatrix},$$

$$E = \begin{vmatrix} & E' & & j \\ & & & k \\ & & & l \\ m & o & -q & p \end{vmatrix}, \quad E' = 2 \begin{vmatrix} a' & c' & -s' \\ d' & f' & -t' \\ g' & i' & -u' \end{vmatrix},$$

$$F = \begin{vmatrix} & & F' & j \\ & & & k \\ & & & l \\ m & o & m^2 + n^2 + o^2 - q^2 & p \end{vmatrix},$$

$$F' = \begin{vmatrix} a' & c' & a'^2 + b'^2 + c'^2 - s'^2 \\ d' & f' & d'^2 + e'^2 + f'^2 - t'^2 \\ g' & i' & g'^2 + h'^2 + i'^2 - u'^2 \end{vmatrix},$$

$$H = \begin{vmatrix} & H' & & j \\ & & & k \\ & & & l \\ n & o & -q & p \end{vmatrix}, \quad H' = 2 \begin{vmatrix} b' & c' & -s' \\ e' & f' & -t' \\ h' & i' & -u' \end{vmatrix},$$

$$I = \begin{vmatrix} & & & j \\ & I' & & k \\ & & & l \\ n & o & m^2+n^2+o^2-q^2 & p \end{vmatrix},$$

$$I' = \begin{vmatrix} b' & c' & a'^2+b'^2+c'^2-s'^2 \\ e' & f' & d'^2+e'^2+f'^2-t'^2 \\ h' & i' & g'^2+h'^2+i'^2-u'^2 \end{vmatrix},$$

$$M = A, \quad N = \begin{vmatrix} & & & -s \\ & A' & & -t \\ & & & -u \\ m & n & o & -q \end{vmatrix},$$

$$O = \begin{vmatrix} & & & a^2+b^2+c^2-s^2 \\ & A' & & d^2+e^2+f^2-t^2 \\ & & & g^2+h^2+i^2-u^2 \\ \mathrm{m} & \mathrm{n} & \mathrm{o} & m^2+n^2+o^2-q^2 \end{vmatrix}.$$

These results could not have been obtained from a Gröbner basis of the polynomial set of all the coefficients of the polynomials in $\mathcal{C}$ with variables being the invariants $A, B, C, D, E, F, G, H, I, J, K, L$. Attempting the Gröbner basis on an Apple Mac Pro server with 6GB of RAM using the computer algebra system Singular (see [12] for an introduction to Singular) gives a "no more memory" error message after 14510 new polynomials have been added to the Gröbner basis. Wu's method provides a constructive method by pseudo-remainders, that is, more predictable and tractable, because one can compute a bound on the number of steps beforehand.

The difficult invariants present in C_1 are expressed as algebraic polynomials in the simple invariants in C_2, C_3, C_4 and C_5, where the new terms in these polynomials (with respect to the $\mathbb{R}^3$ case) are the rightmost ones:

$$\begin{cases} J &= B^2+E^2+H2-A^2+N^2, \\ K &= 2BC+2EF+2HI+2NO, \\ L &= C^2+F^2+I^2+O^2. \end{cases}$$

We find that these invariants are unchanged by any isometry. These invariants of isometries correspond to scalar products of some vectors formed by invariants and powers of points formed by invariants with respect to one of the spheres.

Using geometric invariants represents a very important simplification of Wu's algorithm. Indeed, we have rewritten in a quadratic univariate polynomial in v of the form Jv^2+Kv+L the term in v^2, that had 1411 monomials in the parameters $a, b, c, d, e, f, g, h, i, s, t, u$ into a term J that has only 5 monomials in the simple invariants mentioned above. Moreover, we have rewritten the term in v, that had 9601 monomials in the parameters $a, b, c, d, e, f, g, h, i, s, t, u$ into a term K that has only 4 monomials in the simple invariants mentioned above. Finally, we have rewritten the constant term, that had 24841 monomials in the parameters $a, b, c, d, e, f, g, h, i, s, t, u$ into a term L that has only 4 monomials in the simple

invariants mentioned above. The univariate polynomial of the ascending chain has thus been simplified from a polynomial containing 35853 terms into a polynomial containing only 13 terms using invariants! For the 3D case, we had, respectively, 224, 1080, and 2276 monomials for a total of 3580 monomials. This simplification of Wu's algorithm using geometric and topological invariants can be applied to other geometric or topological problems.

In order to evaluate the Delaunay empty circumsphere predicate for spheres, we need to compute whether the distance between the Voronoi vertex of S_1, S_2, S_3, S_4 and S_5 with coordinates $\begin{pmatrix} x & y & z & w \end{pmatrix}^T$ with respect to the center of S_1 and the sixth hypersphere S_6 with center having coordinates $\begin{pmatrix} x_0 & y_0 & z_0 & w_0 \end{pmatrix}^T$ with respect to the center of S_1 and radius $r_0 + r$ (where r is the radius of the smallest hypersphere among S_1, S_2, S_3, S_4 and S_5) is lower than the (common) distance between $\begin{pmatrix} x & y & z \end{pmatrix}^T$ and S_1, S_2, S_3, S_4 and S_5. However, there are two possible position configurations of the fifth sphere with respect to the Voronoi vertex $\begin{pmatrix} x & y & z \end{pmatrix}^T$: either the fifth sphere does not contain $\begin{pmatrix} x & y & z \end{pmatrix}^T$, or the fifth sphere contains $\begin{pmatrix} x & y & z \end{pmatrix}^T$ (see [1]). The polynomial stating the difference of squared distances between the Voronoi vertex and the sixth hypersphere and between the Voronoi vertex and S_1, S_2, S_3, S_4 and S_5 is $G = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 + (w-w_0)^2 - (r_0 \pm v)^2$. Using repeated division of G with respect to the polynomials of the triangular set $\mathcal{C}$, we get the following univariate (in v) remainders of G by the ascending chain $\mathcal{A}$:
for the Voronoi vertex exterior to the sixth hypersphere, we get $G = G_v v + G_1$, where

$$\begin{aligned} G_v &= \frac{2x_0H}{A} - \frac{2y_0E}{A} + \frac{2z_0B}{A} - 2r_0 + \frac{2w_0N}{M} + 2\frac{NO}{M^2} + \frac{K}{J} - \frac{KN^2}{JM^2} \\ &\quad -\frac{B^2K}{A^2J} + \frac{2BC}{A^2J} - \frac{E^2K}{A^2J} + \frac{2EF}{A^2} - \frac{H^2K}{A^2J} + \frac{2HI}{A^2}, \\ G_1 &= x_0^2 + \frac{2x_0I}{A} + y_0^2 - \frac{2y_0F}{A} + z_0^2 + \frac{2z_0C}{A} - r_0^2 + w_0^2 + \frac{2w_0O}{M} \\ &\quad +\frac{O^2}{M^2} + \frac{L}{J} - \frac{LN^2}{JM^2} - \frac{B^2L}{A^2J} + \frac{C^2}{A^2} - \frac{E^2L}{A^2J} + \frac{F^2}{A^2} - \frac{H^2L}{A^2J} + \frac{I^2}{A^2}; \end{aligned}$$

for the Voronoi vertex interior to the sixth hypersphere, we get $G = G_v v + G_1$, where

$$\begin{aligned} G_v &= \frac{2x_0H}{A} - \frac{2y_0E}{A} + \frac{2z_0B}{A} + 2r_0 + \frac{2w_0N}{M} + 2\frac{NO}{M^2} + \frac{K}{J} - \frac{KN^2}{JM^2} \\ &\quad -\frac{B^2K}{A^2J} + \frac{2BC}{A^2J} - \frac{E^2K}{A^2J} + \frac{2EF}{A^2} - \frac{H^2K}{A^2J} + \frac{2HI}{A^2} \\ G_1 &= x_0^2 + \frac{2x_0I}{A} + y_0^2 - \frac{2y_0F}{A} + z_0^2 + \frac{2z_0C}{A} - r_0^2 + w_0^2 + \frac{2w_0O}{M} \\ &\quad +\frac{O^2}{M^2} + \frac{L}{J} - \frac{LN^2}{JM^2} - \frac{B^2L}{A^2J} + \frac{C^2}{A^2} - \frac{E^2L}{A^2J} + \frac{F^2}{A^2} - \frac{H^2L}{A^2J} + \frac{I^2}{A^2}. \end{aligned}$$

The seven other cases corresponding to the other systems in $\mathcal{II}$ can be computed likewise. In all these cases, the monomials appearing in the polynomials of the triangular system or in the characteristics sets are the same. The expressions of the coefficients J, K and L of the univariate polynomial in v are obviously the same (by construction) in terms of the invariants A, B, C, D, E, F, H and I. Only the coefficients of these monomials in their respective polynomials are different. The algebraic computations needed to get the new expressions are the same except that the addition of the offset parameter to the radius is replaced by the subtraction of the offset parameter from the radius when an expansion is replaced by a retraction. All the formulas can be downloaded from the author's web page `svrfa.spacecenter.dk`.

5. From $\mathbb{R}^N$ to $\mathbb{R}^{N+1}$

The Voronoi vertices satisfy one of the following systems

$$\begin{cases} x_1^2+\ldots+x_N^2+x_{N+1}^2-(v)^2 \\ -2a_{1,1}x_1-\ldots-2a_{1,N+1}x_{N+1}\pm 2r_1v+(a_{1,1}^2+\ldots+a_{1,N+1}^2-r_1^2) \\ -2a_{2,1}x_1-\ldots-2a_{2,N+1}x_{N+1}\pm 2r_2v+(a_{2,1}^2+\ldots+a_{2,N+1}^2-r_2^2) \\ \ldots \\ -2a_{N,1}x_1-\ldots-2a_{N,N+1}x_{N+1}\pm 2r_Nv+(a_{N,1}^2+\ldots+a_{N,N+1}^2-r_N^2) \\ -2a_{N+1,1}x_1-\ldots-2a_{N+1,N+1}\pm 2r_{N+1}v+(a_{N+1,1}^2+\ldots+a_{N+1,N+1}^2-r_N^2) \end{cases}$$

We treat the case where all hyperspheres expand. Using the same Wu's algorithm, we obtain the following characteristic set

$$\begin{cases} C_{N+1,1} & = J_{N+1}v^2+K_{N+1}v+L_{N+1}, \\ C_{N+1,2} & = A_{N+1,1}x_1+B_{N+1,1}v+D_{N+1,1}, \\ C_{N+1,3} & = -A_{N+1,1}x_2+B_{N+1,2}v+D_{N+1,2}, \\ \ldots & \\ C_{N+1,N} & = -A_{N+1,1}x_N+B_{N+1,N}v+D_{N+1,N}, \\ C_{N+1,N+1} & = A_{N+1,1}x_{N+1}+B_{N+1,N+1}v+C_{N+1,N+1}, \end{cases}$$

where

$$A_{N+1,1}=\begin{vmatrix} & & & a_{1,N+1} \\ & A_{N,N} & & \ldots \\ & & & a_{N,N+1} \\ a_{N+1,1} & \ldots & a_{N+1,N} & a_{N+1,N+1} \end{vmatrix}$$

and, for all $I \leqslant N$,

$$B_{N+1,I}=\begin{vmatrix} & & & & & a_{1,N+1} \\ & & B_{N,I} & & & \ldots \\ & & & & & a_{N,N+1} \\ a_{N+1,1} & \ldots & \overline{a_{N+1,I}}\ldots & a_{N+1,N} & -r_{N+1} & a_{N+1,N+1} \end{vmatrix},$$

$$D_{N+1,I}=\begin{vmatrix} & & & & & & a_{1,N+1} \\ & & & D_{N,I} & & & \ldots \\ & & & & & & a_{N,N+1} \\ a_{N+1,1} & \ldots & \overline{a_{N+1,I}} & \ldots & a_{N+1,N} & \sum_{j=1}^{N} a_{N+1,j}^2-r_{N+1}^2 & a_{N+1,N+1} \end{vmatrix},$$

(here $\overline{a_{N+1,I}}$ means that $a_{N+1,I}$ is missing from the row vector $(a_{N+1,1}\ldots,a_{N+1,N})$),

$$B_{N+1,N+1}=\begin{vmatrix} & & -r_1 \\ & A_{N,N} & \ldots \\ & & -r_N \\ a_{N+1,1} & \ldots\quad a_{N+1,N} & -r_{N+1} \end{vmatrix},$$

$$D_{N+1,N+1}=\begin{vmatrix} & & \sum_{j=1}^{N} a_{1,j}^2-r_1^2 \\ & A_{N,N} & \ldots \\ & & \sum_{j=1}^{N} a_{1,j}^2-r_1^2 \\ a_{N+1,1} & \ldots\quad a_{N+1,N} & \sum_{j=1}^{N} a_{N+1,j}^2-r_{N+1}^2 \end{vmatrix}.$$

$J_{N+1}=J_N+B_{N+1,N+1}^2$, $K_{N+1}=K_N+2B_{N+1,N+1}C_{N+1,N+1}$, and $L_{N+1}=L_N+C_{N+1,N+1}^2$.

Finally, we conclude on the degree of the predicate and the precision necessary to compute exactly the results in floating point arithmetic.

PROPOSITION 1. *The algebraic degree of the Delaunay empty hypersphere predicate for hyperspheres in the invariants and the variables defining the $N+2$th hypersphere is $2N$. We need $2N$ times longer bits for the exact computation of the Delaunay empty sphere predicate than the bits used for the invariants and the variables defining the $N+2$th sphere.*

Proof. The Delaunay empty sphere predicate is given by the sign of G. Since the denominator of v is $2J$, the greatest common divider of all the terms in the expansion of G is $A_{N,1}^{2(N-2)}J_N^2$, which is either positive or zero. In the generic case ($A_{N,1}^{2(N-2)}J_N^2 \neq 0$), we can rewrite G as a rational function, where the denominator is $A_{N,1}^{2(N-2)}J_N^2$. Thus, the sign of G is determined only by the sign of the numerator of the preceding rational function. We can see immediately that the degree of this numerator in the invariants and the variables defining the $N+2$th hypersphere is the degree of the monomials $r_0K_NA_{N,1}^{2(N-2)}J_N^2$ or $\left(\sum_{i=1}^{N} x_{0_i}^2-r_0^2\right)A_{N,1}^{2(N-2)}J_N^2$ (where x_0 is the center of the $N+2$th hypersphere), which is $2N$. Bounding all the invariants and the variables defining the $N+2$th hypersphere as in [21], we get that we need $2N$ times longer bits for the exact computation of the Delaunay empty sphere predicate than the bits used for the invariants and the variables defining the fifth sphere.

6. Conclusions

We have proven that the floating point computation of the Delaunay graph of hyperspheres in $\mathbb{R}^N$ requires that the number of digits of the input be $2N$ times higher than the number of digits required for the Delaunay empty sphere predicate. Therefore, this paper contributes to a more precise and faster implementation of the different floating point computation algorithms for the Delaunay graph and the Voronoi diagram of 3D spheres presented in [8, 21, 22], but also for the Delaunay graph and Voronoi diagram of hyperspheres in $\mathbb{R}^N$.

We have generalized the invariants for the Voronoi diagrams of spheres in $\mathbb{R}^3$ presented in [1] to the case of $\mathbb{R}^N$. We found again the same invariants of isometries that correspond to scalar products and determinants (and their minors) of some vectors formed by invariants and powers of points formed by invariants with respect to one of the spheres. The simplification of Wu's algorithm using geometric and topological invariants could be applied to other geometric or topological problems. We have not described the numerical application of these results. However, we have checked these results on some numerical cases. The interested reader can contact the author to get the computer algebra code used to get these results.

Finally, we can conclude that there is an application for the invariants of the Voronoi diagram of 4-dimensional hyperspheres: kinematic Compass, GALILEO, GLONASS, GPS or IRNSS observations in space-time, and that the same methodology that allowed us to get the invariants of the Voronoi diagram of hyperspheres could be applied on ellipses starting from 2D in order to get the invariants of the Voronoi diagram of ellipsoids. This will be the object of future publications.

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ABSOLUTE VORONOI SUMMABILITY OF FOURIER INTEGRALS OF FUNCTIONS OF BOUNDED VARIATION

Liliya Boitsun, Tamara Rybnikova

Abstract. In the present paper, we obtain sufficient conditions imposed on the function $p(t)$, under which the Fourier integral of the function $f(t) \in L(-\infty, \infty)$ is absolutely summable by Voronoi method if the function $\Phi(t) = f(x+t) + f(x-t) - 2f(x)$ is of bounded variation on $(0; +\infty)$.

Key words and phrases: Cesaro method, Fourier integral, summability, function of bounded variation, Voronoi method.

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The functional summability of integrals has been considered by G.F. Voronoi in his work [5]. In the foreign mathematical literature, this method is known as Nerlund method. The history of the summation theory by Voronoi functional method and its applications to trigonometric Fourier integrals, fundamental objects of this theory and review of the achieved results are contained in [4].

Let $p(t)$ be an integrable function on the real semiaxis. If $f(u)$ is integrable in $(0, \infty)$, $S(u) = \int_0^u f(t)\mathrm{d}t$, $P(y) = \int_0^y p(t)\mathrm{d}t \neq 0$, and

$$\tau(y) = \frac{1}{P(y)} \int_0^y P(y-u) f(u)\mathrm{d}u = \frac{1}{P(y)} \int_0^y p(y-u) S(u)\mathrm{d}u,$$

then the integral $\int_0^\infty f(u)\mathrm{d}u$ is said to be absolutely summable by Voronoi method, or $|W, p(y)|$-summable, if $\int_0^\infty |\tau'(y)|\mathrm{d}y < \infty$.

For a function $f(t) \in L(-\infty, \infty)$, its Fourier integral is defined (see [6]) as

$$\frac{1}{\pi} \int_0^\infty \mathrm{d}u \int_{-\infty}^\infty f(t) \cos u(x-t)\mathrm{d}t = \int_0^\infty A(u,x)\mathrm{d}u.$$

We put

$$\phi(t) = \phi(x,t) = f(x+t) + f(x-t) - 2f(x).$$